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Theory and Methodology

Investment and capacity choice under uncertain demand

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Abstract

This paper extends the real options literature by discussing an investment problem, where a firm has to determine optimal investment timing and optimal capacity choice at the same time under conditions of irreversible investment expenditures and uncertainty in future demand. After the project is installed with a certain maximum capacity, this capacity is fixed as an upper boundary to the output and cannot be adjusted later on. It turns out that, in the framework of this once and for all decision, uncertainty in future demand leads to an increase in optimal installed capacity. But on the other hand it causes investment to be delayed to an extent that even small uncertainty makes waiting and accumulation of further information the optimal decision for large ranges of demand. Limiting the capacity which may be installed weakens this extreme effect of uncertainty.

Keywords: Investment analysis; Dynamic programming

1. Introduction

When a firm has the opportunity to invest in a project, its interest is to find the optimal investment strategy using the full freedom of choice that is restricted by technical, administrative or legal constraints. Moreover the firm has to consider that there might be uncertainty in future values of several input quantities concerning this decision. In most cases the investment time is not exogenously fixed. This together with the fact that investment expenditures are at least partially irreversible, makes investment timing one of the main instruments to optimize the firm's strategy. Evaluation of investment opportunities using classical Net Present Value calculation ignores this freedom in timing and that is why more detailed approaches are asked. There are a number of publications – most popular Dixit and Pindyck (1994) – that discuss optimal investment timing in the framework of irreversibility and uncertainty, and point out the parallels between investment opportunities and American call options and, as a consequence, the existence of opportunity costs which fundamentally influence the investment behavior. Furthermore they show the applicability of option pricing methods to determine the value of an option to invest.

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When the project's design offers some additional freedom of choice this real options approach and the underlying standard timing model have to be extended. As capacity choice is typically one of the decisions related closely to investment decisions there are some extensions to the standard timing model published which investigate incremental investment (see Pindyck (1988) for a model investigating incremental investment and Dixit (1995) for an incremental investment model considering scale economies). On one hand these papers present further development of the real options theory but on the other hand they also provide an alternative access to a set of stochastic capacity expansion problems discussed in the past. One early example is Manne (1961) who discusses a stochastic capacity expansion problem with a regenerative property, i.e. there is one capacity decision implemented infinitely often. Bean et al. (1992) and Higle and Corrado (1992) investigate expansion models transforming the stochastic problem to its deterministic equivalent. This approach has a high potential in solving stochastic decision problems and has to be mentioned because of its possible impact on further discussions in real options literature. [1] (For a general survey on capacity expansion literature see Luss (1982).)

The paper presented here is intended to discuss optimal capacity choice in the framework of real options theory, i.e. referring to financial option pricing, the possibility to invest in a project will be evaluated like an American call option. The extensions which are necessary due to the freedom of capacity choice will be derived and they will be displayed by means of an example. In contrast to the models of incremental investment given in Pindyck (1988) and Dixit (1995), I want to discuss a model, where there is only one chance to chose the project's maximum capacity. When the firm decides to exercise its option to invest it has to fix the capacity which will be installed. There is no possibility to adjust the capacity when uncertain parameters – like prices or demand – have changed to unexpected values.

Firms face problems of this kind when capacity-adjustment of a ready built production facility is not possible and the installation of an additional project is out of discussion. There are some reasons why this may be the fact. One is, that you have the unique chance to commit the use of some natural resource, p. ex. to build a hydrostorage plant. Both the maximum electrical power which can be produced as well as the investment expenditures are mainly determined by the height of the water level above the turbine, given by the height of the dam and the height of fall inside the penstock, i.e. the height of the water level corresponds to the capacity. When the plant is completed and the water is dammed up to a certain level, the resource is committed for the lifetime of the plant (i.e. for several decades). The hypothesis of irreversibility of this decision is confirmed by the fact that enlargement of dams is very expensive and therefore rare. (There is no project like this known to me.) Another example is planning a hotel in the center of a city. The capacity choice corresponds to the determination of the number of rooms which shall be established and this decision has to be done during the conception phase of the project. So the maximum capacity is fixed. Due to the fact that attractively situated area is limited there is virtually no possibility to add capacity by building a new hotel when the demand for rooms evolves to an unexpected high level.

The standard problem of investment timing is: As long as the option to invest is alive, the firm has to decide either to keep it alive and wait or to exercise the option, i.e. to pay the investment cost and establish the project. But now the firm has to fix the size of the project and this will cause modification in the investment strategy. As the discussion in this paper will show, uncertainty leads to higher values and higher marginal values of the project. And therefore increasing uncertainty will cause increasing size of the project. But the threshold up to which opportunity costs are positive increases fast with increasing uncertainty, so waiting becomes more valuable and investment is delayed to an extent which is not seen in the standard timing model.

Abel et al. (1996) discuss a two-period model where investment is completely expandable and

[1] This method is not referenced in real options literature. I want to acknowledge the hint given by one of the referees.

reversible within the first period, whereas expandability and reversibility can be restricted within the second period. But in this two-period model the begin of the second period is defined exogenously and with it the moment from that on the expandability is reduced (or vanishes at all). Therefore this kind of two-period models cannot be applied to the problem discussed in this paper.

Section 2 will introduce the model and the assumptions to it, in Section 3 the value of the project and the marginal value of the project are derived. In Section 4 the optimal capacity choice and timing strategy are determined. The sensitivity of the results to the assumptions is discussed and technical details of this discussion are relegated to Appendix A. Section 5 will give a summary of the results.

2. The model

Consider a firm that has the option to invest in a production facility with maximum capacity m, where m has to be fixed during the conception period of the project. That means, the maximum capacity is fixed for the whole lifetime of the facility (assumed to be infinite). The firm faces a demand function of the following form:

$$P = \theta(t) - \delta q, \tag{1}$$

$$P \geqslant 0,$$

where q is the output of the firm, P the price which can be achieved for one unit of output and $\delta = -\mathrm{d}P/\mathrm{d}q$ describes the dependence of the price on the output (so if the firm is a price-taker we have $\delta = 0$). $\theta(t)$ is the demand shift parameter that is assumed to undergo multiplicative geometric Brownian shocks, i.e. it follows a stochastic process of the form

$$\mathrm{d}\theta = \alpha\theta\,\mathrm{d}t + \sigma\theta\,\mathrm{d}z,$$

$$\theta(0) = \theta_0 \geqslant 0, \quad \text{geometric Brownian motion,} \tag{2}$$

where $\mathrm{d}z$ is increment to a Wiener process, α the expected relative drift of θ per unit of time and σ^2 the relative variance per unit of time. That means, the current value of the demand shift parameter is known to the firm but future values are log-normally distributed and the variance of this distribution is increasing as described above. This stochastic process induces uncertainty and, thus risk into the investment problem. Although the expected value of θ shows the familiar exponential behavior ($E(\theta(t)) = \theta_0\,\mathrm{e}^{\alpha t}$), the realization of this process may stay significantly above or below the expected value for long time ranges.

To determine the profit flow π the firm receives when the project is installed, I assume the marginal production costs c' to be a function of the project's size but to be constant with respect to the output ($c' = c'(m)$).

$$\pi(\theta, m, q) = [P(\theta, q) - c'(m)]q, \quad 0 \leqslant q \leqslant m. \tag{3}$$

So the profit flow π, which will be used to determine the value of the project, follows a stochastic process, too. The investment costs I that have to be payed for installation of a production facility are a function of the facility's maximum capacity $I = I(m)$. They are assumed to be sunk costs and have the general form

$$I(m) = bm^{\gamma}, \quad \gamma \leqslant 1, \tag{4}$$

i.e. the marginal investment costs are decreasing with increasing installed capacity.

The firm's task now is to observe this system and to decide either to wait or to invest and fix the size of the project. As the stochastic process θ is not explicitly a function of time, the current level of the demand shift parameter is the only system information on which the investment decision is based.

3. Value and marginal value of the project

When the firm decides to exercise the option to invest and to install a project with capacity m it has to pay $I(m)$ and receives a project worth V. To evaluate this decision it is necessary to know the value of the project, which is a function of the current level of the demand shift parameter θ, and the installed capacity m. The capacity m is fixed over the project's lifetime and the future motion of

θ is a realization of the process defined by (2). So the derivation of V follows the standard approach (see Pindyck, 1991; Dixit and Pindyck, 1994) using Dynamic Programming.

The firm is assumed to choose its output q to maximize V.

$$V(\theta, m) = E \max_q \int_0^\infty \pi(\theta(t+\tau), m, q(t+\tau))\, \mathrm{e}^{-r\tau}\, \mathrm{d}\tau, \tag{5}$$

where r is the individual discount rate of the firm ($r > \alpha, r > 0$).

The optimal output q is found by maximizing the profit flow π at every point. i.e. to adjust the output instantaneously with the objective of maximizing the current profit flow when the level of the demand shift parameter θ has changed. Using (3) gives

$$q(\theta, m) = \begin{cases} 0, & \theta < c', \\ (\theta - c')/2\delta, & c' \leqslant \theta < c' + 2\delta m, \\ m, & c' + 2\delta m \leqslant \theta. \end{cases} \tag{6}$$

We get the boundary values $\theta_1 = c'$ and $\theta_2 = c' + 2\delta m$, that define three ranges for θ: $R_1 = [0, \theta_1)$, $R_2 = [\theta_1, \theta_2)$, $R_3 = [\theta_2, \infty)$

$$\pi(\theta, m, q(\theta, m)) = \pi(\theta, m) = \begin{cases} \pi_1(\theta, m) = 0, & \theta \in R_1, \\ \pi_2(\theta, m) = (\theta - c')^2/4\delta, & \theta \in R_2, \\ \pi_3(\theta, m) = (\theta - c')m - \delta m^2, & \theta \in R_3. \end{cases} \tag{7}$$

After substitution of Eq. (7) into Eq. (5) the calculation of V will be done by the use of Dynamic Programming:

$$V(\theta, m) = \pi(\theta, m)\, \mathrm{d}t + \mathrm{e}^{-r\, \mathrm{d}t} E(V(\theta, m) + \mathrm{d}V(\theta, m)). \tag{8}$$

This leads (applying Ito's Lemma, serial expansion and neglecting terms of the order o(dt)) to the nonhomogenous differential equation

$$\frac{1}{2}\sigma^2\theta^2 \frac{\partial^2 V}{\partial \theta^2}(\theta, m) + \alpha\theta \frac{\partial V}{\partial \theta}(\theta, m) - rV + \pi(\theta, m) = 0 \tag{9}$$

which has to be satisfied by the solution for values of θ inside the three ranges R_j defined by Eq. (6).

The boundary conditions will be stated later. The solution can be written as

$$V(\theta, m)|_{\theta \in R_j} = V_j(\theta, m) = A_{j,1}(m)\theta^{\beta_1} + A_{j,2}(m)\theta^{\beta_2} + \overline{V}_j(\theta, m), \quad j = 1 \ldots 3, \tag{10}$$

with β_1 and β_2 solving the quadratic equation

$$\frac{1}{2}\beta(\beta - 1)\sigma^2 + \beta\alpha - r = 0, \quad \beta_1 > 1, \quad \beta_2 < 0. \tag{11}$$

$A_{j,k}$ are functions of the capacity m. Applying the boundary conditions will determine them. $\overline{V}_j$ is a particular solution of the differential equations (9) with $\pi = \pi_j$. Eq. (12) gives particular solutions $\overline{V}_{1,2,3}$:

$$\overline{V}_j = \begin{cases} 0, & j = 1;\ (\theta \in R_1), \\ \dfrac{1}{4\delta}\left[\dfrac{\theta^2}{r - 2\alpha - \sigma^2} - \dfrac{2c'\theta}{r - \alpha} + \dfrac{c'^2}{r}\right], & j = 2;\ (\theta \in R_2), \\ \dfrac{\theta m}{r - \alpha} - \dfrac{\delta m^2 + c'm}{r}, & j = 3;\ (\theta \in R_3). \end{cases} \tag{12}$$

As the demand shift parameter θ is volatile, it is able to cross the boundaries θ_j freely and as a consequence to this the solution has to satisfy the boundary conditions

$$V(\theta_i^-, m), = V(\theta_i^+, m), \quad i = 1, 2, \tag{13a}$$

$$V_\theta(\theta_i^-, m), = V_\theta(\theta_i^+, m), \quad i = 1, 2. \tag{13b}$$

Eqs. (13a) and (13b) define four of the six parameters $A_{j,k}$ of Eq. (10). Another can be determined by using a characteristic of the geometric Brownian motion (2): 0 is an absorbing point of the process θ. As there is no profit flow at this point we get the condition

$$V(0, m) = 0. \tag{13c}$$

The last boundary condition concerns the behavior of the solution for big values of θ. For $\theta \in R_3$ growing to big values, the probability that θ will cross the boundary θ_2 in the near future decreases. Therefore one expects, that the effect of the existence of the other regions vanishes, more precisely

$$\lim_{\theta\to\infty} V = E\left(\int_0^\infty \pi_3(\theta,m)\,\mathrm{e}^{-rt}\,\mathrm{d}t\right)$$
$$= \frac{\theta m}{r-\alpha} - \frac{\delta m^2 + c' m}{r} = \overline{V}_3. \tag{13d}$$

(The integral exists because $r > \alpha$ (5), selecting the solution that does not contain speculative bubbles gives the expression shown in Eq. (13d).) $\overline{V}_3$ can be called the fundamental component of solution V_3 (Dixit and Pindyck, 1994) that would give the full solution of Eq. (9) if R_3 covers the whole definition range $[0,\infty)$ of θ. This leads to the interpretation of the remaining part of V_3 $(A_{3,1}(m)\theta^{\beta_1} + A_{3,2}(m)\theta^{\beta_2})$ describing the deviation of V_3 from the fundamental component due to the restriction of R_3, i.e. due to the possibility that θ will move across the boundaries $\theta_{1,2}$.

Notice, that for region R_2 $E\left(\int_0^\infty \pi_2(\theta,m)\mathrm{e}^{-rt}\,\mathrm{d}t\right)$ leads formally to $\overline{V}_2$ (see Abel and Eberly, 1994, Lemma 1), but interpreting $\overline{V}_2$ as fundamental component, which describes the project's value if R_2 covers $[0,\infty)$, is restricted to cases where $r - 2\alpha - \sigma^2 > 0$. Otherwise there is no convergence of the integral.

In cases, where $r - 2\alpha - \sigma^2 = 0$, no analytical solution for $\overline{V}_2$ can be found, but as the value of the project V is continuous in r, α and σ (substitute Eq. (7) into Eq. (5)) $V(\theta, m)$ can be defined statically as $\lim_{\sigma^2\to r-2\alpha} V(\theta,m)$.

Fig. 1 shows the value V of the project for several values of σ. One can see that increasing uncertainty in demand causes higher value V.

So far this is the standard way to calculate the value of a project under uncertainty. Now we have to consider that the maximum capacity of the project m is free to choose. Assuming the firm decided to invest at a level θ of the demand shift parameter, we have to answer the question, which capacity should be installed. To do this, the marginal value of the project has to be determined.

$$\frac{\mathrm{d}V(\theta,m)}{\mathrm{d}m} = \frac{\partial V(\theta,m)}{\partial c'}\frac{\mathrm{d}c'}{\mathrm{d}m} + \frac{\partial V(\theta,m)}{\partial m}. \tag{14a}$$

There is no derivation with respect to θ, because $\mathrm{d}\theta/\mathrm{d}m = 0$.

$$\left.\frac{\mathrm{d}V(\theta,m)}{\mathrm{d}m}\right|_{\theta\in R_j} = \frac{\mathrm{d}A_{j,1}(m)}{\mathrm{d}m}\theta^{\beta_1} + \frac{\mathrm{d}A_{j,2}(m)}{\mathrm{d}m}\theta^{\beta_2} + \frac{\partial\overline{V}_j(\theta,m)}{\partial c'}\frac{\mathrm{d}c'}{\mathrm{d}m} + \frac{\partial\overline{V}_j(\theta,m)}{\partial m}. \tag{14b}$$

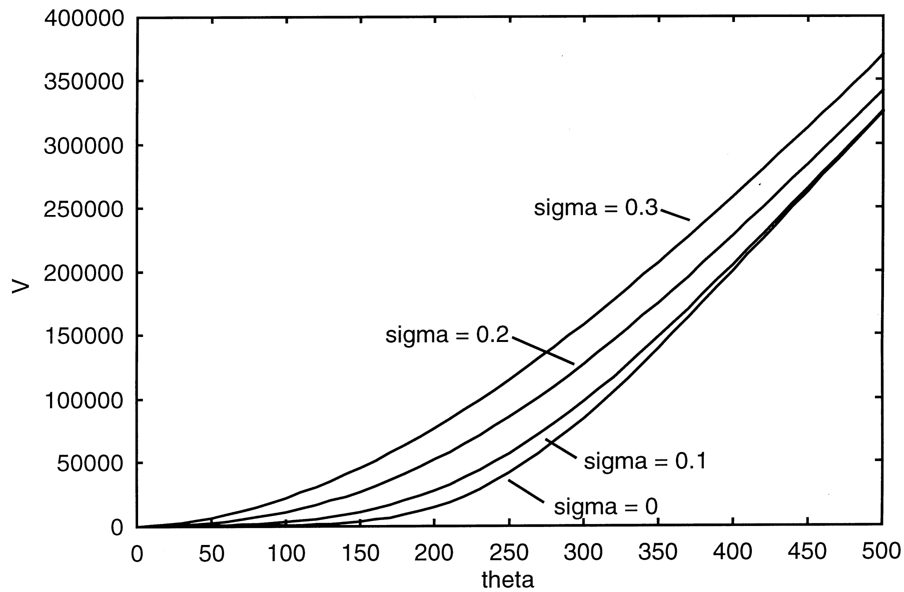


Fig. 1. Value $V(\theta, m)$ of the project as function of demand shift parameter θ. Calculated with $r = 0.1$, $\alpha = 0.02$, $c' = 200$, $\delta = 1$, $m = 100$, $\sigma = 0$, 0.1, 0.2, 0.3.

Write this as

$$v(\theta,m)|_{\theta\in R_j} = v_j(\theta,m) = a_{j,1}(m)\theta^{\beta_1} + a_{j,2}(m)\theta^{\beta_2} + \overline{v}_j(\theta,m). \tag{14c}$$

$\overline{v}_j$ can be determined by derivation of Eq. (12):

$$\overline{v}_j = \begin{cases} 0, & j=1;\ (\theta\in R_1), \\ \dfrac{1}{4\delta}\left[-\dfrac{2\theta}{r-\alpha}+\dfrac{2c'}{r}\right]\dfrac{\mathrm{d}c'}{\mathrm{d}m}, & j=2;\ (\theta\in R_2), \\ \dfrac{\theta}{r-\alpha}-\dfrac{2\delta m+c'}{r}-\dfrac{m}{r}\dfrac{\mathrm{d}c'}{\mathrm{d}m}, & j=3;\ (\theta\in R_3). \end{cases} \tag{14d}$$

For calculation of the $a_{j,k}$ comparative statics are applied to the boundary conditions (13a)–(13d). Investigation of Eq. (13a) gives

$$\frac{\mathrm{d}}{\mathrm{d}m}[V(\theta_i^-,m) - V(\theta_i^+,m)] = 0,$$

$$v(\theta_i^-,m) - v(\theta_i^+,m) + \underbrace{\left(\frac{\partial V}{\partial\theta}(\theta_i^-,m) - \frac{\partial V}{\partial\theta}(\theta_i^+,m)\right)}_{=0}\frac{\mathrm{d}\theta_i}{\mathrm{d}m} = 0, \quad i=1,2,$$

$$v(\theta_i^-,m) = v(\theta_i^+,m), \quad i=1,2. \tag{15a}$$

Comparative static analysis of Eq. (13b) gives

$$\frac{\mathrm{d}}{\mathrm{d}m}\left[\left(\frac{\partial V}{\partial\theta}(\theta_i^-,m) - \frac{\partial V}{\partial\theta}(\theta_i^+,m)\right)\right] = 0.$$

Considering the mixed derivations $\partial^2 V/\partial\theta\,\partial m$, $\partial^2 V/\partial m\,\partial\theta$, $\partial^2 V/\partial\theta\,\partial c'$, and $\partial^2 V/\partial\theta\partial c'$ to be continuous inside the ranges R_j leads to

$$\frac{\partial}{\partial\theta}v(\theta_i^-,m) - \frac{\partial}{\partial\theta}v(\theta_i^+,m) = \frac{\partial^2 V}{\partial\theta^2}(\theta_i^+,m)\frac{\mathrm{d}\theta_i}{\mathrm{d}m} - \frac{\partial^2 V}{\partial\theta^2}(\theta_i^-,m)\frac{\mathrm{d}\theta_i}{\mathrm{d}m}, \quad i=1,2. \tag{15b}$$

Eq. (15a) says that the value matching condition is also valid for the marginal value of the project, i.e. v is a continuous function across the boundaries θ_1 and θ_2. But (see Eq. (15b)) it is not necessarily smooth at these boundaries; there might be a kink at the transition from one region to the next. As some lengthy but plain derivations (which shall not be presented) show, $\partial^2 V/\partial\theta^2$ defined by Eqs. (10)–(12) and Eqs. (13a)–(13d) is continuous at $[0,\infty)$. More generally this characteristic can be deduced using the results of Feynman and Kac (see Karatzas and Shreve, 1988, Ch. 4) together with the fact that the profit flow π is a continuous function of θ, so condition (15b) can be simplified to a 'smooth pasting' condition:

$$\frac{\partial}{\partial\theta}v(\theta_i^-,m) = \frac{\partial}{\partial\theta}v(\theta_i^+,m), \quad i=1,2. \tag{15b$'$}$$

Eqs. (15a) and (15b′) define four of the six parameters $a_{j,k}$, the remaining two parameters are determined by inspection of the conditions (13c) and (13d).

$$v(0,m) = 0, \tag{15c}$$

$$\lim_{\theta\to\infty}\left[v(\theta,m) - \left(\frac{\theta}{(r-\alpha)} - \frac{2\delta m+c'}{r} - \frac{m}{r}\frac{\mathrm{d}c'}{\mathrm{d}m}\right)\right] = 0. \tag{15d}$$

Fig. 2 shows the marginal value v of the project as a function of the demand shift parameter θ. The marginal value is increasing with increasing uncertainty and, as prescribed by the boundary conditions (15a) and (15b′), it is continuous and smooth at the boundaries θ_1 ($=200$) and θ_2 ($=400$). Note, that the marginal project is only utilized when $\theta > \theta_2$. Otherwise the output is reduced below maximum capacity m. Due to the prospect of future utilization the expected drift and the volatility of θ create a positive value of the marginal project even if $\theta \leqslant \theta_2$. The asymptotic behavior of v for $\theta\to\infty$ is characterized by the term $\theta/(r-\alpha)$ of Eq. (15d) which is the net present value of selling one (i.e. the marginal) unit of output at the price θ (θ following the process (2); the existence of the boundary θ_2 can be neglected for large values of θ), whereas the asymptotic behavior of the project's value V is characterized by $(\theta m)/(r-\alpha)$ of Eq. (13d) which is the value of selling the limit of m units at a price of θ. (Compare Fig. 1 with Fig. 2: The ratio of V to v converges to $m = 100$.)

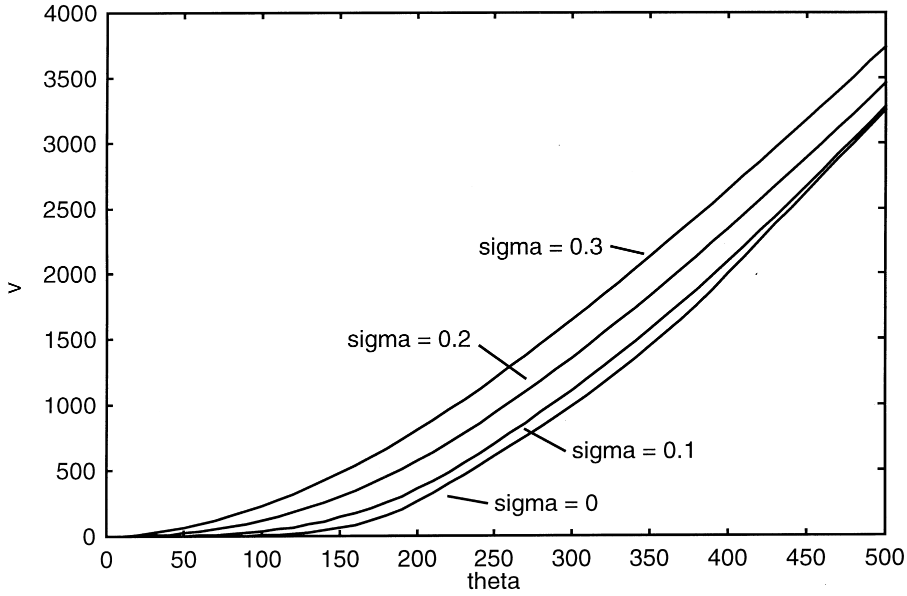


Fig. 2. Marginal value $v(\theta, m)$ of the project as function of demand shift parameter θ. Uncertainty leads to an increase in the marginal value of the project. Parameters: $r = 0.1$, $\alpha = 0.02$, $\delta = 1$, $c' = 200$, $dc'/dm = -1$, $m = 100$, $\sigma = 0, 0.1, 0.2, 0.3$.

4. The value of the option to invest and the optimal investment strategy

As mentioned in the Section 1, there are two instruments available for the firm to adjust its strategy. First: To subdivide the definition range of the demand shift parameter θ into regions, where the firm will invest immediately ('investment regions'), and regions, where the firm will not invest but investigate the system and accumulate further information ('waiting regions'). Second: If θ is inside the investment region, the firm has to fix the maximum capacity m it will install. The value, the opportunity to invest offers to the firm, is determined by these two decisions. And it is of in the firms' interest to choose its strategy to maximize the value F of this option. So the firm's problem can be written as

$$F(\theta) = \max\{e^{-r\,dt}E(F(\theta) + dF(\theta)),\\ \max_m[V(\theta, m) - I(m)]\}. \tag{16a}$$

The outer maximization corresponds to the general decision whether the project should be installed at the level θ of the demand shift parameter or not. If the firm decides to wait (for an infinitesimal span of time dt) the value the firm holds is just the discounted expected value of the option after dt passed by. This is represented by the first argument of the maximization. The second argument of the outer maximization represents the value of the project the firm receives when it decides to invest reduced by the investment cost. The inner maximization says that the firm, when it decides to invest, will chose the project's capacity to maximize the value it will receive.

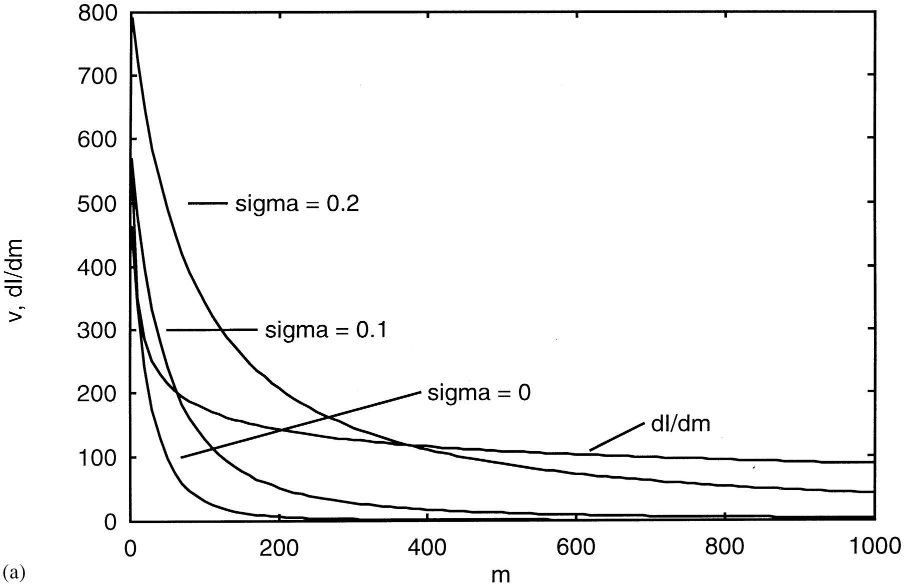
As a first step consider the firm ignores its freedom in investment timing but decides either to invest now or never. So it has just to find the optimal maximum capacity m^* that should be installed ($m^* = 0$ corresponds to the decision not to invest). Then the value of the option is

$$F(\theta) = \max_m[V(\theta, m) - I(m)] \tag{16b}$$

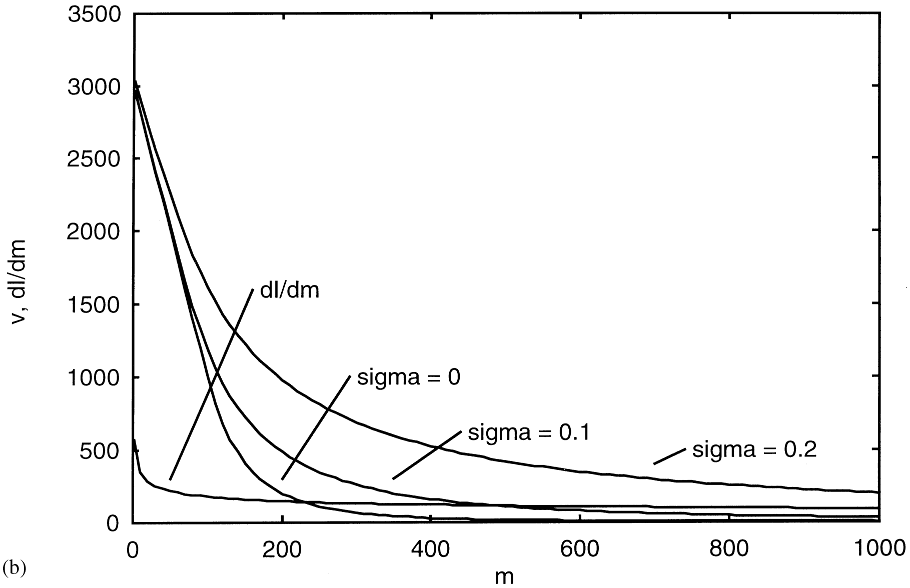
and the condition for an inner extremum

$$\frac{\partial}{\partial m}(V(\theta, m) - I(m)) = v(\theta, m) - \frac{dI(m)}{dm} = 0. \tag{17}$$

This is the common condition, that at the optimal choice of the maximum capacity m^* the marginal



(a)



(b)

Fig. 3. Marginal value $v(\theta, m)$ of the project as function of maximum capacity m. The optimal choice of m is at the intersection of v and $\mathrm{d}I/\mathrm{d}m$. (a): At $\theta = 200$. Calculated with $r = 0.1$, $\alpha = 0.02$, $\delta = 1$, $c' = 200$, $\mathrm{d}c'/\mathrm{d}m = 0$, $\sigma = 0$, 0.1, 0.2, and marginal investment costs $\mathrm{d}I/\mathrm{d}m$, calculated with $b = 1000$, $\gamma = 0.7$. Notice that without uncertainty ($\sigma = 0$) there is no intersection of v and $\mathrm{d}I/\mathrm{d}m$ ($v < \mathrm{d}I/\mathrm{d}m$): If there is no freedom in timing, only the existence of uncertainty in demand will lead to investment. (b): At $\theta = 400$. Calculated with $r = 0.1$, $\alpha = 0.02$, $\delta = 1$, $c' = 200$, $\mathrm{d}c'/\mathrm{d}m = 0$, $\sigma = 0$, 0.1, 0.2, and marginal investment costs $\mathrm{d}I/\mathrm{d}m$, calculated with $b = 1000$, $\gamma = 0.7$. Now even without uncertainty there exists an optimal maximum capacity $m^* > 0$. Notice that for $\sigma = 0.2$ $v - \mathrm{d}I/m$ is even positive at $m = 1000$.

value of the project has to be equal to the marginal investment costs. But here the marginal value of the project is influenced by the uncertainty related to the future values of θ. With the results of Section 3 one is able to find solutions of Eq. (17) for different values of θ numerically.

Fig. 3(a) and (b) shows the marginal value of the project as a function of the maximum capacity m for two levels of the demand shift parameter θ and the marginal investment costs. It can be seen, that higher values of σ cause higher marginal value of the project and slower approach to zero. As the optimal capacity choice $m^*(\theta)$ is at the intersection of v and $\mathrm{d}I/\mathrm{d}m$ (see Eq. (17)), uncertainty leads to significant higher values of installed capacity. Furthermore it turns out that, ignoring freedom in timing, at small values of θ there is no investment without a certain level of uncertainty. These facts are demonstrated by Fig. 4, which shows the optimal choice m^* defined by Eq. (17) as a function of θ for 3 different levels of σ.

After this calculation we know the net payoff $V(\theta, m^*(\theta)) - I(m^*(\theta))$ the firm receives in case of immediate investment. Fig. 5 shows the high dependence of this payoff on σ as a consequence of the optimal behavior referring to the capacity choice. This fact is the special feature of this model with one unique chance to choose the maximum capacity and will influence the optimal timing strategy to a great extent.

To find this optimal timing strategy is the task of the following considerations. The question is: at which levels of θ is it optimal to exercise the ‘real option’ (investment region) and at which levels of θ is it optimal to keep the option alive (waiting region)? As the firm is free to choose its investment strategy, it can divide the definition range of θ into any number of intervals being waiting and investment regions alternating. But Fig. 5 shows that $V(\theta, m^*(\theta)) - I(m^*(\theta))$ is increasing with θ and that is, together with the fact that the stochastic process θ is a geometric Brownian motion, a sufficient condition that the optimal timing strategy has the following simple form (see Dixit and Pindyck, 1994, Ch. 3, Appendix B): For θ up to a threshold θ^* it is optimal to keep the real option alive, for $\theta \geqslant \theta^*$ it is optimal to invest immediately. When θ exceeds θ^* and the firm will exercise its option to invest, the value of this option is known due to the calculations above. For

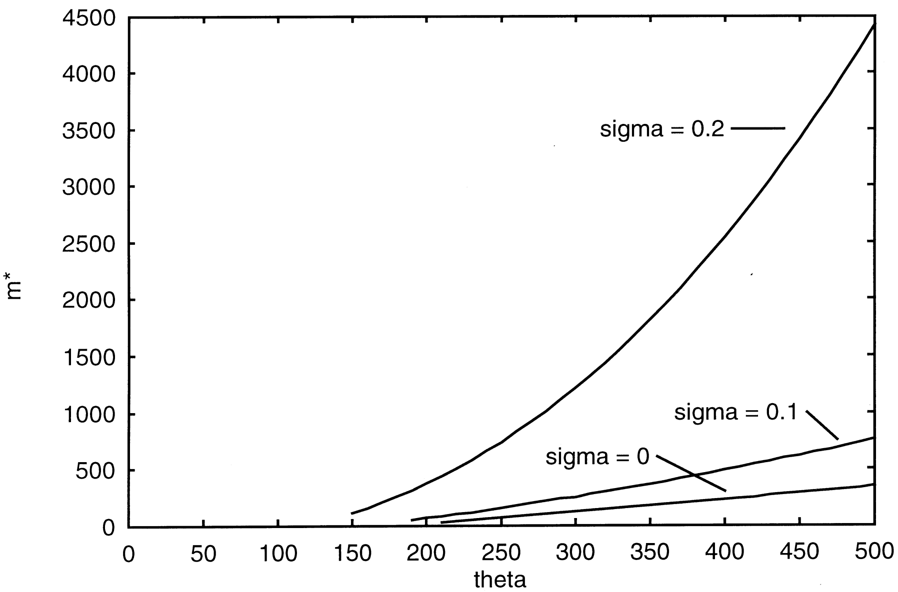
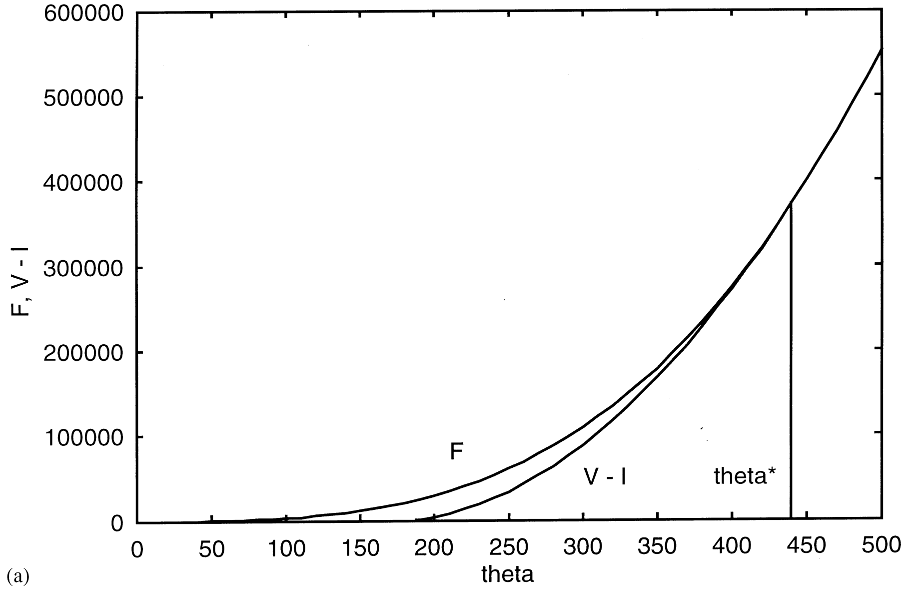
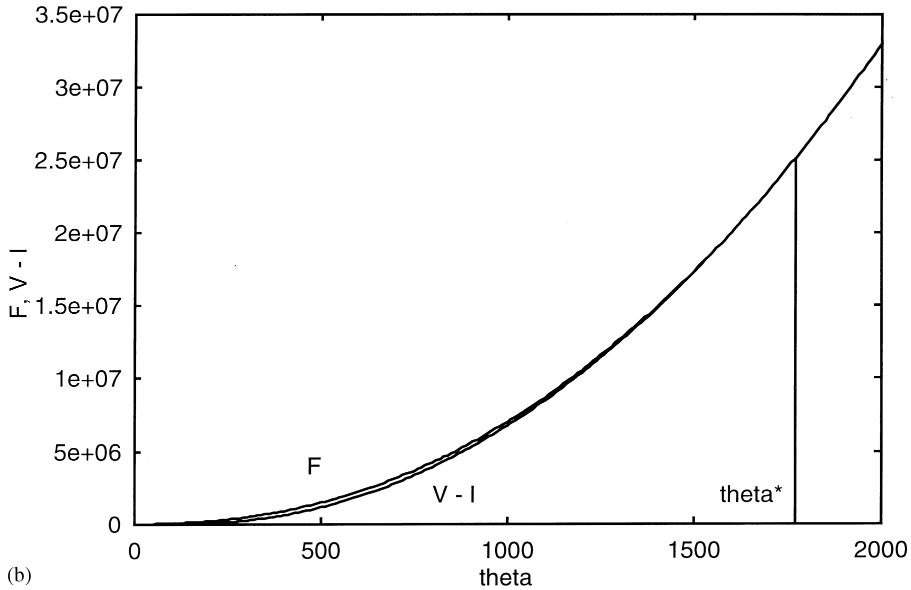


Fig. 4. The optimal choice of maximum capacity $m^*(\theta)$. $m^*(\theta)$ is monotonically growing with θ. Uncertainty causes significant higher values of optimal installed maximum capacity. Parameters: $r = 0.1$, $\alpha = 0.02$, $\delta = 1$, $c' = 200$, $\mathrm{d}c'/\mathrm{d}m = 0$, $b = 1000$, $\gamma = 0.7$, $\sigma = 0, 0.1, 0.2$.



(a)



(b)

Fig. 5. The value of the option F to invest together with the net payoff $V - I$. (a): For $\sigma = 0.1$. The threshold θ^* between waiting and investment region is at 440. (b): For $\sigma = 0.2$. The threshold θ^* between waiting and investment region is at 1770. Common parameters: $r = 0.1$, $\alpha = 0.02$, $\delta = 1$, $c' = 200$, $dc'/dm = 0$, $b = 1000$, $\gamma = 0.7$.

$\theta < \theta^*$ F is represented by the first argument of the outer maximization in Eq. (16a):

$$F(\theta) = e^{-r\,dt} E(F(\theta) + dF(\theta)). \tag{16c}$$

Applying Ito's Lemma, serial expansion and neglecting terms of the order o(dt) gives the following differential equation that is equal to Eq. (9) but without the nonhomogenous term:

$$\frac{1}{2}\sigma^2\theta^2\frac{d^2F}{d\theta^2}(\theta) + \alpha\theta\frac{dF}{d\theta}(\theta) - rF = 0. \quad (18)$$

Eq. (18) has a similar but simpler solution than Eq. (9):

$$F(\theta) = C_1\theta^{\beta_1} + C_2\theta^{\beta_2}, \quad (19)$$

where β_1 and β_2 are the positive and negative solutions of Eq. (11).

One boundary condition is again (see condition (15c)) based on the fact that θ is stable at 0 and as the value V of the project is 0 there

$$F(0) = 0. \quad (20a)$$

And therefore: $C_2 = 0$.

To avoid arbitrage profits at the transition from the waiting to the investment region one can follow the second boundary condition

$$C_1\theta^{*\beta_1} = V(\theta^{*+}, m^*(\theta^{*+})) - I(m^*(\theta^{*+})). \quad (20b)$$

The conditions (20a) and (20b) would be sufficient to fix the solution (19), if the threshold θ^* is given. But θ^* has to be chosen to maximize the value of the option. That is

$$\max_{\theta^*}\{C_1\} = \max_{\theta^*}\left\{\frac{1}{\theta^{*\beta_1}}[V(\theta^*, m^*(\theta^*)) - I(m^*(\theta^*))]\right\}. \quad (20c)$$

Condition (20c) is equivalent to the commonly used smooth pasting condition which says that the value F of the option has to be smooth at the transition from the waiting to the investment region.

With Eqs. (19), (20a), (20b) and (20c) the parameter C_1 (and with it the value F of the option in the waiting region) and the threshold θ^* can be determined. Fig. 5 shows the value of the option together with the net payoff $V(\theta, m^*(\theta)) - I(m^*(\theta))$ the firm receives in case of investment. The optimal threshold θ^* between waiting and investment region is at the point where the two curves meet smoothly. This threshold is increasing enormously with σ, as a consequence of the high dependence of the optimal choice of maximum capacity m^* on σ. Table 1 shows θ^* and $m^*(\theta^*)$ for 3 values of σ.

Table 1
Optimal threshold θ^* between waiting and investment region and optimal capacity choice $m^*(\theta^*)$ at this point

σ	θ^*	$m^*(\theta^*)$
0.05	346	210
0.1	440	592
0.2	1770	93 232

Values of m lower than $m^*(\theta^*)$ will never be installed.

(Calculations for $\sigma = 0.3$ give a positive slope of C_1 even at $\theta = 3400$ but m^* at this point exceeds 10^8.)

If the demand shift parameter θ is lower than θ^*, the value of the real option F exceeds the net payoff $V - I$, and the firm will not invest but it will design a project with maximum capacity $m^*(\theta^*)$ and wait until θ reaches the threshold θ^*. At this point the option to invest will be exercised. At levels of θ exceeding θ^* the firm will install a project with maximum capacity $m^*(\theta)$ immediately. Since $m^*(\theta)$ is monotonically growing, $m^*(\theta^*)$ is the smallest capacity which will be installed at all. So the project's size is simply exploding with increasing uncertainty on the one hand, but on the other hand, this project will hardly be installed because the waiting region is growing at the same time. The probability that there is investment in the near future vanishes with increasing uncertainty. Even small values of uncertainty in demand cause waiting and investigating the system the optimal strategy for large regions of demand (see Table 1).

This peculiar attribute of the model can be interpreted as a consequence of the once and for all characteristic of the investment decision but is, obviously, affected by the particular choice of parameters and by model assumptions. The following will discuss the sensitivity of the results to some of these suppositions. The assumed existence of scale economies (see Eq. (4)) is a fact that encourages the construction of large sized projects. Raising the parameter γ of the cost function (4) reduces this advantage of large projects and as a consequence it damps the optimal size of the project. But due to the rise in investment costs the delay of the investment is reinforced (p. ex. shifting γ from 0.7 to 0.8 changes the entries of the last row of Table 1 to: $\theta^* = 2335$, $m^* = 49\,337$; $\gamma = 0.9$ yields: $\theta^* = 3136$, $m^* = 30\,647$). Although the numerical result is affected by the choice of γ, the quality remains: Uncertainty leads to high capacities but the actual investment is delayed.

The enormous delay of investment which is observed is advanced by the absence of an upper limit for the maximum capacity that can be installed. Changing this assumption and defining a limit for m damps both the size of the project and the delay of the investment (see Table 2, m is limited to 1000).

See Fig. 6 where the maximum capacity that is optimal to install is plotted as function of θ for several values of σ. For the example presented (see Table 2): When σ exceeds 0.1195, installation of capacity below the limit of 1000 is not worth to be discussed at all. However, increasing uncertainty will increase the threshold θ^* from which on it is optimal to exercise the option and to install the capacity of 1000.

Table 2
Optimal threshold θ^* between waiting and investment region and optimal capacity choice $m^*(\theta^*)$ at this point when the maximum capacity is limited to 1000

σ	θ^*	$m^*(\theta^*)$
0.05	346	210
0.1	440	592
0.1195	498	993
0.2	560	1000
0.3	632	1000

Limiting the maximum capacity m that can be installed weakens the effect of uncertain demand on the critical threshold θ^*.

A core assumption to the model is the definition of the demand shift parameter θ following a geometric Brownian motion (2), so θ is log-normally

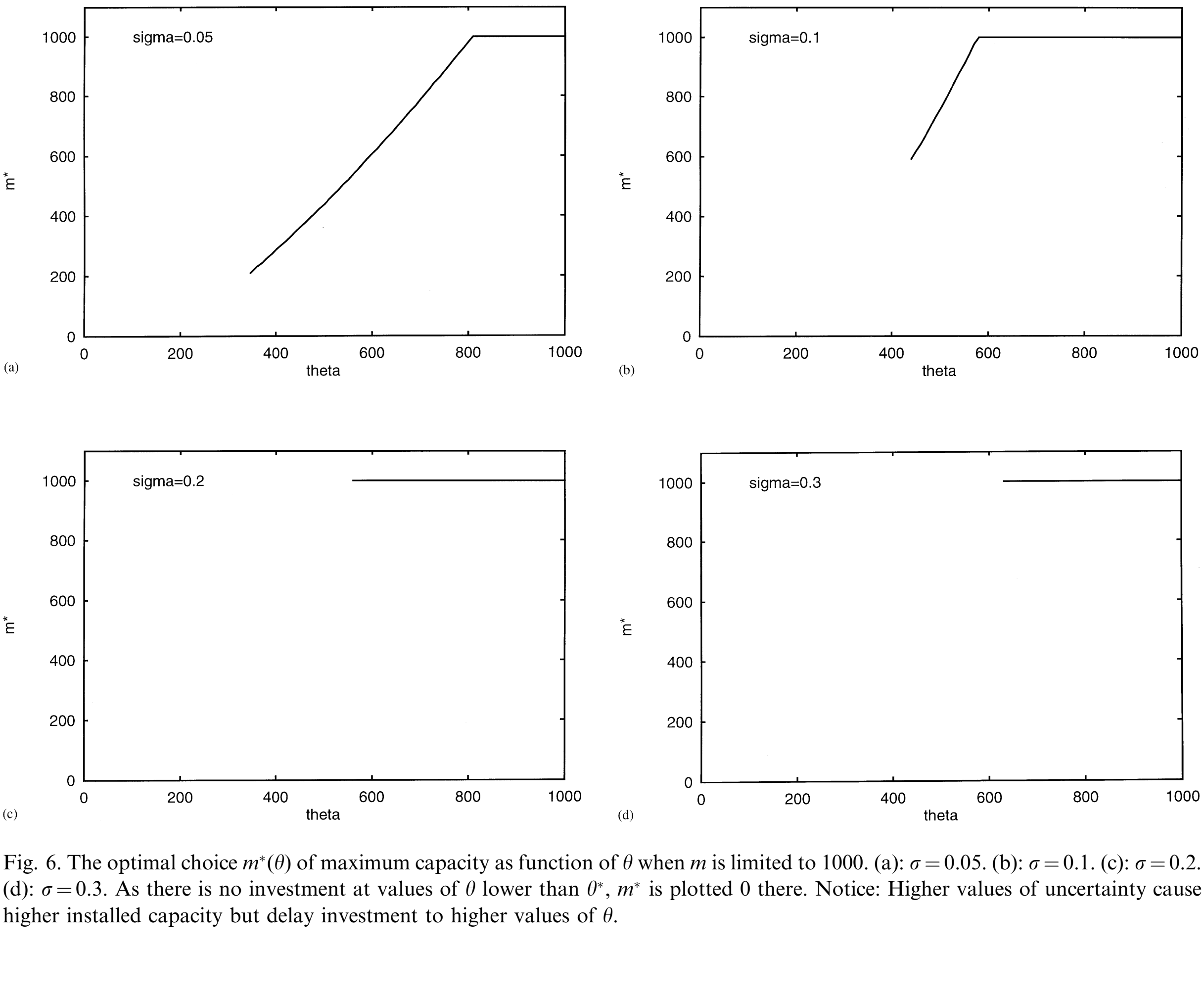


Fig. 6. The optimal choice $m^*(\theta)$ of maximum capacity as function of θ when m is limited to 1000. (a): $\sigma = 0.05$. (b): $\sigma = 0.1$. (c): $\sigma = 0.2$. (d): $\sigma = 0.3$. As there is no investment at values of θ lower than θ^*, m^* is plotted 0 there. Notice: Higher values of uncertainty cause higher installed capacity but delay investment to higher values of θ.

distributed and its expected value increases exponentially with time. An alternative is to define θ as simple Brownian motion with drift which causes linear behavior of the expected value. The investigation of the model applying this alternative assumption (see Appendix A) shows that exponential behavior of θ is in fact responsible for the enormous increase in optimally installed capacity m^* and for the exceptional delay of the decision (represented by θ^*) which are described above. But the quality of the result is the same even for linear behavior of the demand shift parameter θ: Increasing demand uncertainty leads to an increase in the amount of optimal capacity and to a delay of the actual investment to receive additional demand information – although both effects are weakened.

5. Summary

This paper extends the real options literature (Dixit and Pindyck, 1994) by combining optimal investment timing with optimal capacity choice at the same time. Once installed, the project is completely unexpandable. Numerical investigations show that the optimal installed capacity increases very much with uncertainty. And this has the consequence that investment is delayed to high demand ranges to an extent which cannot be seen in the standard investment timing model. So even small uncertainty in demand makes waiting and accumulating further information the optimal decision for large ranges of demand.

Looking at the results of a model with incremental investment (Pindyck, 1988), we see a contrary investment strategy. Uncertainty reduces the optimal amount of installed capacity. The tendency to 'oversize' a project, which was investigated in this paper, can be interpreted as an effect of the once and for all decision. This leads to an interesting modification that could be applied to the current model: One could inspect a chain of possible projects where the size of each project has to be fixed at the time of its installation. Realizing a project one receives not only the value of the project but also the real option to invest in the remaining chain. When the number of projects in this chain is growing, the effect of the once and for all characteristic of the capacity choice vanishes and the effect of higher flexibility due to smaller projects should dominate the decision and will cause investment in smaller sized projects and reduce the threshold up to which waiting is optimal (see Manne (1961) for a particular example).

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Appendix A

In the paper the demand shift parameter θ is assumed to follow a geometric Brownian motion (2) so that θ is log-normally distributed and the expected value increases exponentially with time. To investigate the sensitivity of the results to this assumption Appendix A discusses the model with θ following a simple Brownian motion with drift (A.2). So θ is normally distributed and the expected value increases linearly with time. To keep the description brief and clear, equation (A.x) of Appendix A always corresponds to equation (x) in the paper.

The demand function is given by Eq. (1), the demand shift parameter θ follows the stochastic process

$$\mathrm{d}\theta = \alpha\,\mathrm{d}t + \sigma\,\mathrm{d}z, \quad \text{Brownian motion.} \tag{A.2}$$

The determination of the profit flow and the general considerations on the project's value are not concerned by the particular form of θ therefore Eqs. (3)–(8) are unchanged. The differential equation which has to be satisfied by the value function is

$$\frac{1}{2}\sigma^2\frac{\partial^2 V}{\partial\theta^2}(\theta,m) + \alpha\frac{\partial V}{\partial\theta}(\theta,m) - rV + \pi(\theta,m) = 0. \tag{A.9}$$

The solution of the homogenous part of this equation is now the exponential function in contrast to the power function in the paper. We get

$$V(\theta,m)|_{\theta\in R_j} = V_j(\theta,m) = A_{j,1}(m)\,\mathrm{e}^{\lambda_1\theta} + A_{j,2}(m)\,\mathrm{e}^{\lambda_2\theta} + \overline{V}_j(\theta,m), \quad j = 1\ldots 3, \tag{A.10}$$

where λ_1 and λ_2 are the roots of the characteristic equation

$$\frac{1}{2}\lambda^2\sigma^2 + \lambda\alpha - r = 0, \quad \lambda_1 > 0, \;\; \lambda_2 < 0, \tag{A.11}$$

and the particular solution $\overline{V}_j$ is given by

$$\overline{V}_j = \begin{cases} 0, & j=1;\;\; (\theta\in R_1), \\ \dfrac{1}{4\delta}\left[\dfrac{(\theta-c')^2}{r} + \dfrac{2\alpha(\theta-c')+\sigma^2}{r^2} + \dfrac{\alpha^2}{r^3}\right], & j=2;\;\; (\theta\in R_2), \\ \dfrac{(\theta-c')m-\delta m^2}{r} + \dfrac{\alpha m}{r^2}, & j=3;\;\; (\theta\in R_3). \end{cases} \tag{A.12}$$

The derivation of the value of the marginal project v follows the same idea as sketched in the paper, the arguments for 'value matching' and smooth pasting conditions (15a) and (15b′) are valid. The value function of the real option to invest in the project changes also to an exponential function and has the form

$$F(\theta) = C_1\,\mathrm{e}^{\lambda_1\theta}. \tag{A.19}$$

The investigation of this model with linear behavior of the demand shift parameter θ gives results of the same quality as they are obtained with θ following a geometric Brownian motion, whereas the effect is weakened: Increasing uncertainty leads to a higher amount of optimal installed capacity and to a delay of the actual investment. Table 3 displays a numerical example.

Table 3
Optimal threshold θ^* between waiting and investment region and optimal capacity choice $m^*(\theta^*)$ at this point, θ following a simple Brownian motion

σ	θ^*	$m^*(\theta^*)$
40	379	222
80	458	412
160	609	998

The values of α ($=8$) and σ are chosen so that drift and volatility of Eqs. (2) and (A.2) are comparable at $\theta = 400$.

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